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AN ALGORITHM AND ERROR ESTIMATES FOR THE VARIATIONAL BOUNDARY-ELEMENT METHOD IN ELASTICITY THEORY†

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In the variational boundary-element method (VBEM) proposed in [1–3], the problems are posed for boundary functionals and generalized Treftz functionals and the solutions are approximated by discrete boundary potentials (DBP). In this paper, the VBEM is used to solve some problems of elasticity theory. Some problems of the order of the boundary-element (BE) approximations and the structure of the matrices of systems of discrete boundary equations (DBE) are considered. The "influence" function is constructed using the fundamental solutions. Error estimates are obtained using the functionals of the dual problems.

1. VBEM ALGORITHMS are considered as they apply to the minimization of boundary functionals (BF) of the planar and spatial problems of linear isotropic elasticity theory, assuming that these problems satisfy the Korn inequality [4], or that a finite-energy solution exists (for problems in an infinite domain [4]). As an example, we consider the second problem of elasticity theory (with given stresses on the boundary) in the elastic domain $G\subset E^{(m)}$, $m=2$, 3, with a sufficiently smooth boundary S (that satisfies the conditions of the trace theorem). The corresponding variational problem involves minimizing the quadratic functional [4] (the mass forces are ignored)

$$F_G(\mathbf{u})=2\int_G W(\mathbf{u})\,dG-2\int_S \mathbf{g}^{(\nu)}\mathbf{u}\,ds$$

on feasible displacement vector functions $\mathbf{u}(x)$, $x\in\bar{G}$, where $2W(\mathbf{u})$ is the quadratic form of linear isotropic elasticity theory and $\mathbf{g}^{(\nu)}(y)$, $y\in S$ is the vector of given stresses along the outer normal $\boldsymbol{\nu}$. We know [4] that the solution of the problem of minimizing $F_G(\mathbf{u})$ exists apart from an arbitrary rigid displacement.

The variational problem is reduced to the boundary in the following way. Assume that the vector $\mathbf{u}$ satisfies the equilibrium equation $\mathbf{Au}(x)=0$, $x\in G$. Then by Betti's formula [4] we obtain the equality

$$2\int_G W(\mathbf{u})\,dG=\int_S \mathbf{t}^{(\nu)}(\mathbf{u})\,\mathbf{u}\,ds$$

where $\mathbf{t}^{(\nu)}(\mathbf{u})$ is the vector of the normal surface stresses. Thus, the problem $\min_{\mathbf{u}}F_G(\mathbf{u})$ can be replaced by the equivalent problem for the BF

$$\min_{\mathbf{u}\in D}F_S(\mathbf{u}),\quad D=\{\mathbf{u}:\mathbf{Au}(x)=0,\,x\in G\}\tag{1.1}$$

$$F_S(\mathbf{u})=\int_S \mathbf{t}^{(\nu)}(\mathbf{u})\,\mathbf{u}\,ds-2\int_S \mathbf{g}^{(\nu)}\mathbf{u}\,ds$$

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If $\min_{\mathbf{u}\in D} F_S(\mathbf{u}) = F_S(\mathbf{u}_0)$, then the vector $\mathbf{u}_0$ satisfies the boundary variational equation

$$\int_S \mathbf{t}^{(\mathbf{v})}(\mathbf{u}_0)\mathbf{u}\,ds - \int_S \mathbf{g}^{(\mathbf{v})}\mathbf{u}\,ds = 0, \quad \forall \mathbf{u} \in D \tag{1.2}$$

The VBEM algorithm for problem (1.1) essentially reduces to the BE approximation of Eq. (1.2).

2. The BE approximation of regular variational problems (i.e. variational problems without any singularities that affect the approximation: for instance, corner points near which the solution increases rapidly, etc.) is conveniently obtained by isoparametric BE approximations [5] (see also [1]), in which the approximation nodes are identical for the boundary and the required solution and BEM basis functions of the same order are used for approximation. To solve the problems that follow, we will use these approximations in the form

$$y_n^{(i)}(\eta) = \sum_{k=1}^{K} y_{nk}^{(i)} \psi_k(\eta), \quad i = 1, \ldots, m \tag{2.1}$$

$$u_n^{(i)}(\eta) = \sum_{k=1}^{K} U_{nk}^{(i)} \psi_k(\eta), \quad \forall n \tag{2.2}$$

where $y_{nk}^{(i)}$ are the Cartesian (global) coordinates of the nodes partitioning the boundary S, k is the node index of the boundary elements Δs_n, $U_{nk}^{(i)}$ are the nodal values of the components of the displacement vector $\mathbf{u}$, ψ_k are the BEM basis functions and η is the local coordinate of the BE points.

Let $S_\Delta = \cup \Delta s_n$, $n = 1, \ldots, N$ be the discrete boundary approximating S (or $S_\Delta \equiv S$), and G_Δ the domain bounded by S_Δ; we assume that the approximation (2.2) satisfies the BE compatibility condition, which in our case signifies continuity of the global interpolation function across the boundary between the elements and is achieved by equality of the nodal values of the required solution at the common nodes of adjacent elements [1]. Then

$$y_\Delta = \sum_{n=1}^{N} y_n(\eta), \quad \mathbf{u}_N(y_\Delta) = \sum_{n=1}^{N} \mathbf{u}_n(\eta) \tag{2.3}$$

respectively, are the parametric equation of S_Δ and the continuous function that approximates the solution at the points $y_\Delta \in S_\Delta$ with the normal $\boldsymbol{\nu}_\Delta$.

In what follows, we will use the standard integral representation [6] of a sufficiently smooth function in G_Δ by its boundary values $\mathbf{u}_N(y_\Delta)$, $\partial_{\nu_\Delta}\mathbf{u}_N(y_\Delta)$, $y_\Delta \in S_\Delta$. This representation, originally developed in potential theory [6], has been written for problems of elasticity theory in the form [7]

$$\alpha_N(x_\Delta) = -\frac{1}{2}\int_{S_\Delta} \mathbf{t}^{(\mathbf{v}_\Delta)}\Big(\sum_{j=1}^{m} v^{1j}\Big)\mathbf{u}_N(y_\Delta)\,ds(y_\Delta) +$$

$$+\frac{1}{2}\int_{S_\Delta}\sum_{j=1}^{m} v^{1j}\mathbf{t}^{(\mathbf{v}_\Delta)}\big(\mathbf{u}_N(y_\Delta)\big)\,ds(y_\Delta), \quad x_\Delta \in G_\Delta \tag{2.4}$$

Here $\{v^{ij}\}$ is the tensor of fundamental solutions of the equation of elasticity theory (the Somigliana tensor). The representation (2.4) has been proved [6, 7] for a piecewise-smooth boundary S_Δ. We know that $\mathbf{A}\boldsymbol{\alpha}_N(x_\Delta) = 0$ for all points x_Δ inside and outside S_Δ. The integral representation (2.4) suggests alternative formulations of BE approximations: if Green's tensor of the first problem of statics (or Green's tensor of the second problem of statics) is taken as the fundamental solution, then the corresponding integral in (2.4) vanishes (such BE approximations are considered in detail in [1, 2]).

The representation (2.4) assumes that the approximations (2.2) are sufficiently smooth: for the existence of the second integral we should have at least $\mathbf{u}_N \in W_2^2(S_\Delta)$ [$W_2^2(S_\Delta)$ is the Sobolev class of functions that are continuously differentiable at the points S_Δ, and the approximation (2.1) is also assumed to be sufficiently smooth]. In each of the cases described above, the vector functions $\alpha_N(x_\Delta)$, $x_\Delta \in G_\Delta$, $\forall N$ are admissible functions of the finite-dimensional variational problem $\min F_{S_\Delta}(\alpha_N)$, $\alpha_N \in D_\Delta$ approximating problem (1.1), because they satisfy the equilibrium equation in G_Δ. We then apply the Ritz process [1], which produces a discrete variational equation approximating equation (1.2):

$$\sum_{n=1}^{N} \int_{U\Delta s_n} \mathbf{t}^{(\mathbf{v}_n)} \Big(\sum_{i=1}^{m} u_n^{(i)} \Big) \psi_l \, ds(y_n) -$$
$$- \sum_{n=1}^{N} \int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k=1}^{K} g_{nk}^{(i)} \psi_k \psi_l \, ds(y_n) = 0, \quad l = 1, \ldots, K \tag{2.5}$$

($\mathbf{v}_n(\eta)$ is the outer normal at the points of Δs_n); in (2.5) for a given vector function $\mathbf{g}^{(\nu)}$ we have used a BE approximation of the form (2.2), where $g_{nk}^{(i)}$ are the nodal values of the components of the discrete function $\mathbf{g}^{(\mathbf{v}_n)}$ and integration is over the union of the BEs $\cup \Delta s_n$, for which k is a common node. Equation (2.5) is essentially the Ritz system of DBE (in contracted form) for the nodal values $U_{nk}^{(i)}$; successively writing the equations for each node k, we obtain the sums Σ_n with non-zero integral coefficients for the contribution of the BEs for which k is a common node. Thus, the matrix of the Ritz system is banded (the bandwidth depends on the order of the BE, see below) and symmetric, so that

$$\int_{\Delta s_n} \partial_n^{(i)} \psi_k \psi_l \, | J_n | \, ds_n = \int_{\Delta s_n} \partial_n^{(i)} \psi_l \psi_k \, | J_n | \, ds_n, \; \partial_n^{(i)} \equiv \frac{\partial}{\partial y_n^{(i)}} \tag{2.6}$$

The transition from (2.5) to the DBE system involves [1, 2] BE approximation of the vector of boundary stresses [7]

$$\mathbf{t}^{(\mathbf{v})}(\mathbf{u}) = 2\mu \partial_\mathbf{v} \mathbf{u} + \lambda (\mathbf{v} \cdot \operatorname{div} \mathbf{u}) + \mu (\mathbf{v} \times \mathbf{rot}\, \mathbf{u})$$

on the approximations (2.2) of the displacement vector. This BE approximation can be written at the points of Δs_n in the form [1, 2]

$$\mathbf{t}^{(\mathbf{v}_n)} \Big(\sum_{i=1}^{m} u_n^{(i)} \Big) = \sum_{i=1}^{m} \sum_{k=1}^{K} U_{nk}^{(i)} T_n \psi_k, \quad \forall n \tag{2.7}$$

where T_n is a scalar operator, whose form is established from the componentwise expression for the vector $\mathbf{t}^{(\nu)}(\mathbf{u})$:

$$T_n \psi_k = 2\mu \sum_{i=1}^{m} \partial_n^{(i)} \psi_k l_n^{(i)} + \lambda \sum_{i=1}^{m} \partial_n^{(i)} \psi_k \sum_{i=1}^{m} l_n^{(i)} +$$
$$+ \mu \sum_{\substack{i, j \\ (i \neq j)}}^{m} (\partial_n^{(j)} - \partial_n^{(i)}) \psi_k \, (l_n^{(i)} + l_n^{(j)}), \quad \partial_n^{(i)} \equiv \frac{\partial}{\partial y_n^{(i)}} \tag{2.7a}$$

where $y_n^{(i)}(\eta)$ is the BE approximation (2.1). Since $\psi_k = \psi_k(\eta)$, the derivatives $\partial_n^{(i)} \psi_k$ in (2.7a) are evaluated by the rules of differentiation of a compound function; the direction cosines $l_n^{(i)}$ of the normal $\mathbf{v}_n$ are computed by transforming the differentials of the area (the length) of the BEs Δs_n from the local coordinate system (η) to the global coordinate system (y_n) [5, 8]. We define this transformation as

$$ds(y_n) = | J_n | \, d\eta \Rightarrow s_n = \int | J_n | \, d\eta = \operatorname{diam} \Delta s_n$$

Here $|J_n|$ is the determinant of the Jacobi matrix $[J_n]$ of the transformation (2.1). By [5, 8], we have in general ($m = 2, 3$)

$$|J_n| = \left\{\sum_{i=1}^{m} d_{in}^2\right\}^{1/2}, \quad l_n^{(i)} = d_{in}|J_n|^{-1}$$

where d_{in} are the minors of the matrix $[J_n]$, which are defined in terms of the derivatives $\partial_{\eta j} y_n^{(i)}$, i, $j = 1, 2, 3$.

Having determined from (2.5) the nodal values $U_{nk}^{(i)}$, $i = 1, \ldots, m$, we can represent the "Ritz" BE approximations of the solution of problem (1.1) and the equivalent (see Sec. 1) boundary-value problem with given stresses on S in the form

$$\bar{\mathbf{u}}_N \equiv \alpha_N(x_\Delta) = \sum_{n=1}^{N}\sum_{k=1}^{K} U_{nk}\alpha_{nk}(x_\Delta), \quad x_\Delta \in G_\Delta \tag{2.8}$$

where, by (2.4), the "influence functions" of the kth node of the nth BE are defined as superpositions of scalar potentials with density concentrated on the BE (see also [1, 2])

$$\alpha_{nk} = -\frac{1}{2}\int_{\Delta s_n} \mathbf{t}^{(\nu_n)}\left(\sum_{j=1}^{m} v^{1j}\right)\psi_k |J_n| ds_n(\eta) +$$

$$+\frac{1}{2}\int_{\Delta s_n}\sum_{j=1}^{m} v^{1j} T_n \psi_k |J_n| ds_n(\eta), \quad \forall n \tag{2.9}$$

For instance, in the St Venant problem of torsion of an isotropic homogeneous rod, which in terms of the scalar warping function of the rod cross-section reduces to a variational problem of the form (1.1) on the set of harmonic functions, the functions (2.9) are defined as harmonic DBP (of a double or a simple layer); with linear BE approximation of the elliptical contour of the rod cross-section, we obtain the analytical expression

$$\alpha_{nk} = \frac{1}{2\pi}\ln|x_\Delta - y_{nk}|, \quad x_\Delta = \sum_{i=1}^{2} x_\Delta^{(i)}, \quad y_{nk} = \sum_{i=1}^{2} y_{nk}^{(i)}$$

where $y_{nk}^{(i)}$ are the Cartesian coordinates of the partition nodes. The functions α_{nk} define the level lines of the warping of the rod cross-section under torsion and correspond to the classical function that describes the source or sink surface [9] (depending on the sign of the nodal values). A linear combination of the products of the nodal values, which are obtained by BE approximation of the variational problem (1.1), and of the functions α_{nk}, $k = 1, 2$, $n = 1, \ldots, N$, produces a semi-analytical (numerical-analytical) solution of the St Venant problem [2].

For vector problems of elasticity theory, (2.9) is computed using the row vector of the Somigliana tensor $\mathbf{v}^{1j} = (v^{11}, v^{12}, v^{13})$. We know that this vector is the solution (for $x \neq y$) of the homogeneous Lamé equation: for the plane problem, the components of the vector are given by [7]

$$v_n^{11} = c_0[c_1 r_n^{-1} + (x^{(1)} - y_n^{(1)})^2 r_n^{-3}], \quad (x^{(1)}, x^{(2)}) \in G_\Delta$$

$$v_n^{12} = c_0(x^{(1)} - y_n^{(1)})(x^{(2)} - y_n^{(2)}) r_n^{-3}, \quad (y_n^{(1)}, y_n^{(2)}) \in S_\Delta \tag{2.10}$$

$$r_n = \left\{\sum_{i=1}^{2}(x^{(i)} - y_n^{(i)})^2\right\}^{1/2}, \quad c_0 = [16\pi\mu(1-\sigma)]^{-1}, \quad c_1 = 3 - 4\sigma$$

where $y_n^{(i)}(\eta)$, $\eta \in \Delta s_n$ is the BE approximation (2.1). A simple analysis show that the computation of α_{nk} with linear BE approximation (2.1) using (2.10) reduces to taking integrals of the form

$$\int \eta^q B^{-p} d\eta, \quad B = a\eta^2 + b\eta + c, \quad q = 0, 1, 2, \quad p = 1/2, 3/2, 5/2,$$

where the coefficients a, b, c depend on the coordinates $x_\Delta^{(i)} \in G_\Delta$, $y_{nk}^{(i)} \in S_\Delta$. These integrals are computed analytically [10, pp. 67–68] and contain a logarithmic function of the distance r_n, as well as the power functions r_n^{-p}. Thus, the "influence functions" α_{nk} in the BE approximations (2.8) are defined $\forall x_\Delta \in G_\Delta$. The domain G_Δ may be unbounded, in which case the simple layer potential decreases at infinity as $O(r_n^{-1})$ and the double layer potential decreases as $O(r_n^{-2})$. The final formulas for α_{nk} are fairly complicated, because the BE approximation of the vector $\mathbf{t}^{(\nu_n)}(\Sigma u_n^{(i)})$ is not simple [see (2.7), (2.7a)].

3. Accuracy estimation of the BE approximations $\{\mathbf{u}_N\}$ [see (2.8)] of the solution of the finite-dimensional variational problem $\min F_{s_\Delta}(\mathbf{u}_N)$, $\mathbf{u}_N \in D_\Delta$ reduces to estimating the error of the BE approximations $\mathbf{u}_N(y_\Delta)$ at the points $y_\Delta \in S_\Delta$, because at the points $x_\Delta \in G_\Delta$ the feasible functions of the problem exactly satisfy the differential equation of the boundary-value problem. These estimates can be obtained using the *a posteriori* error estimates of the approximate solutions of the dual variational problems for the BF of linear elasticity theory [11], which, as it applies to BE approximation estimates, take the form

$$\| \mathbf{u}_{0\Delta} - \mathbf{u}_N \|_{1/2, S_\Delta} \leqslant c_+ \Delta(\mathbf{u}_N), \quad c_+ > 0 \tag{3.1}$$

$$\| \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_{0\Delta}) - \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) \|_{-1/2, S_\Delta} \leqslant c_- \Delta(\mathbf{u}_N), \quad c_- > 0$$

$$\Delta(\mathbf{u}_N) = \{c_\Delta^{-1}[F_{S_\Delta} - \Phi_{S_\Delta}]\}^{1/2}, \quad c_\Delta > 0$$

(here the constants c_+, c_- are independent of N). In (3.1), $\mathbf{u}_{0\Delta}$ is the boundary value of the solution u_0 of the problem (1.1) at the points $y_\Delta \in S_\Delta$; F_{S_Δ}, Φ_{S_Δ} are the BFs of the dual problems [11], which are computed, respectively, on the BE approximations $\mathbf{u}_N$, $\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$ [see (2.3) and (2.7)]; the constant c_Δ can be determined from the estimate [11]

$$2 \int_{G_\Delta} W(\mathbf{u})\, dG_\Delta \geqslant c_\Delta \| u \|_{1, G_\Delta}^2, \quad \forall \mathbf{u} \in W_2^1(G_\Delta)$$

The $W_2^{\mp 1/2}(S_\Delta)$ norms used in (3.1) are well-defined because by construction [see (2.3)] $\mathbf{u}_N \in W_2^1(S_\Delta)$. Note that in a numerical experiment (see the problems below) it is sufficient to establish that as the partition S_Δ into boundary elements is made finer, the difference $F_{S_\Delta} - \Phi_{S_\Delta}$ decreases. This can be done [11] using the equality

$$F_{S_\Delta} - \Phi_{S_\Delta} = 2 \int_{S_\Delta} \mathbf{u}_N [\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) - \mathbf{g}_N^{(\nu_\Delta)}]\, ds_\Delta > 0 \tag{3.2}$$

where $\mathbf{g}_N^{(\nu_\Delta)}$ is the BE approximation of the given vector function $\mathbf{g}^{(\nu)}$ [see (2.5)], and the integral on the right-hand side is evaluated as

$$I_{S_\Delta} = \sum_{n=1}^{N} \int_{\Delta s_n} \sum_{i=1}^{m} u_n^{(i)} \left[\mathbf{t}^{(\nu_n)} \left(\sum_{i=1}^{m} u_n^{(i)} \right) - g^{(\nu_n)} \right] ds_n \tag{3.3}$$

[here we use the approximations (2.2) and (2.7)]. Formula (3.3) is essentially the approximation of (1.2) at the points of S_Δ. By virtue of the convergence of BE approximations [3], we have $I_{s_\Delta} \to 0$ as $N \to \infty$ [11].

These *a posteriori* estimates do not provide any information about the order of the approximation error, i.e. about the rate of convergence of the approximations $\mathbf{u}_N \to \mathbf{u}_0$ as $N \to \infty$ ($\operatorname{diam} \Delta s_n \to 0$). This information is provided by *a priori* error estimates. Here we can use the *a priori* estimates of the Bubnov–Galerkin finite-element (FE) approximations [12]. Such estimates have been constructed for approximate solutions of second-order elliptical boundary-value problems.

Convergence of BE approximations based on the integral representation (2.4) has been established [3] for the case when S_Δ is a finite union of Lyapunov surfaces, which corresponds to a compatible union of the BEs Δs_n, as used in the conformal FEM [12]. For such approximations, the construction of an *a priori* estimate of the "global" FEM (BEM) interpolation reduces to the construction of an error estimate of the "local" interpolation on a single finite element [12], using the estimate of the approximation error of a sufficiently smooth function by FEM interpolants in some norm. Thus, the error of the BE approximations $\{\mathbf{u}_N\}$ [see (2.3)] can be estimated using the following bound [12]:

$$\| \mathbf{u}_{0\Delta} - \mathbf{u}_N \|_{1, S_\Delta} \leqslant c d^r \| \mathbf{u}_{0\Delta} \|_{r+1, S_\Delta}, \quad c > 0$$

where $d = \operatorname{diam} \Delta s_n$, $r \geqslant 1$ is the local order of approximation accuracy, $\|\cdot\|_{1,S_\Delta}$ is the norm in the Sobolev class of vector functions $W_2{}^1(S_\Delta)$; the constant c is independent of d. In the $L_2(S_\Delta)$ norm, the maximum order of interpolation accuracy is higher [12].

4. We considered a generalization of the Flamant problem [13] for an elastic half-plane loaded by normal pressure along a second-order curve $y^{(2)} = \frac{1}{2}(y^{(1)})^2 R^{-1} \equiv S$ at the points $y = (y^{(1)}, y^{(2)}) \in S$; the pressure function was taken [14] in the form of the solution of the problem of a punch with a surface described by the curve S penetrating (without friction) into the half-plane. This problem was posed in displacement terms as the variational problem (1.1) without mass forces with the supplementary conditions

$$\mathbf{t}^{(\nu)}(\mathbf{u}(y)) = 0, \quad y \in [-\infty, \infty] \setminus s, \quad \boldsymbol{\tau}(\mathbf{u}(y)) = 0, \quad y \in [-\infty, \infty]$$

where $\boldsymbol{\tau}(\mathbf{u})$ is the tangential stress vector. For the BE approximation of the problem, we used linear approximations of the form (2.1), (2.2) and BE approximations (2.8). The functions (2.9) were computed using the sum $v_n{}^{11} + v_n{}^{12}$, $\forall n$ [see (2.10)]. The Ritz system [see (2.5)] is constructed from the DBE, which are formed according to the "pattern" $\forall n = 1, \ldots, N$, $i, j = 1, 2$:

$$-\left(\mu + \frac{1}{2}\lambda\right) U_{n1}^{(i)} (f_n^{(i)} + f_{n-1}^{(i)}) - \frac{1}{2}\mu U_{n1}^{(i)} (f_n^{(j)} + f_{n-1}^{(j)}) - \frac{1}{4}(\mu+\lambda) U_{n1}^{(j)} -$$

$$-\left(\mu + \frac{1}{2}\lambda\right) U_{(n+1)1}^{(i)} (f_n^{(i)} + f_{n+1}^{(i)}) - \frac{1}{2}\mu U_{(n+1)1}^{(i)} (f_n^{(j)} + f_{n+1}^{(j)}) - \frac{1}{4}(\mu+\lambda) U_{(n+1)1}^{(j)} =$$

$$= \frac{1}{3} p_{n1}^{(i)} (Y_n^{(j)} - Y_{n-1}^{(j)}) + \frac{1}{3} p_{(n+1)1}^{(i)} (Y_{n+1}^{(j)} - Y_n^{(j)})$$

Here we have implemented the approximation (2.7), using the notation

$$f_n^{(i)} = \frac{1}{2} Y_n^{(j)} (Y_n^{(i)})^{-1}, \quad |J_n| = \left\{ \sum_{i=1}^{2} (Y_n^{(i)})^2 \right\}^{1/2}$$

$$\partial_\eta y_n^{(i)} = Y_n^{(i)} = \frac{1}{2} (y_{n2}^{(i)} - y_{n1}^{(i)}), \quad i = 1, 2, \quad \forall n$$

where $y_{nk}^{(i)}$ are the coordinates of the nodes $k = 1, 2$, and $p_{nk}^{(i)}$ are the nodal values of the given function $p_i^{(\nu)}(y^{(1)})$ (see [14, p. 65]). The numerical implementation was considered for the following special case of material characteristics: modulus of elasticity $E = 10^5$, Poisson's ratio $\sigma = 0.3$, Lamé constants $\lambda = 0.5769E$, $\mu = 0.3846E$ and radius of curvature of S, $R = 1$. The coordinates of the nodes partitioning the curve S on one side of the axis of symmetry $y^{(2)}$ (for $N = 6$) and the nodal values of the displacements obtained by solving the DBE system (for $E = 1$) are given below:

$10^4 \times y_{n1}^{(1)}$:	0;	335;	671;	1005;	1338;	1670;	2000
$10^4 \times y_{n1}^{(2)}$:	0;	5.6;	22.5;	50.5;	89.5;	139;	200
$10^4 \times U_{n1}^{(1)}$:	0;	−45;	116;	−179;	253;	−320;	763
$10^4 \times U_{n1}^{(2)}$:	−1453;		6.45;	−14.5;	20.5;	−24;	26.5; 0

Analysis of the results leads to the following conclusions. There are no horizontal displacements on the axis of symmetry $y^{(2)}$, while the vertical displacements are small. As we move away from the $y^{(2)}$ axis, the components $U_{nk}^{(2)}$ decrease and the components $U_{nk}^{(1)}$ conversely increase due to the increase in the shear strains. The "deformed" coordinates of the partition nodes are given by $\tilde{y}_{nk}^{(i)} = y_{nk}^{(i)} + U_{nk}^{(i)} \times 10^5$, $i = 1, 2$.

A numerical experiment was carried out, evaluating the sum (3.3) for $N = 6, 12, 24$: $\Sigma_6 \cong 0.22$, $\Sigma_{12} \cong 0.205$, $\Sigma_{24} \cong 0.175$. The difference (3.2) thus decreases relatively slowly as the BE partition is refined, because we are essentially approximating a piecewise-linear function $\mathbf{g}^{(\nu_n)}$ [see (2.5)] by a piecewise-constant function $\mathbf{t}^{(\nu_n)}(\Sigma u_n^{(i)})$ [see (2.7)]. In the duality algorithm [15], to solve the prototype contact problem based on the variational problem (1.1), we have used second-order BE approximations, which produce more-accurate approximate stresses. Note also that with first order BE approximations, the $K_N \times K_N$ square matrix of the DBE system (K_N is the number of nodes in S_Δ) is banded with bandwidth 4; with second-order BE approximations, the bandwidth is 6.

We considered the problem of displacements for the points of a spherical surface with zero boundary conditions for the displacement vector on meridional (or annular) lines and acted upon by a normal surface load. Note that these conditions on a set of three-dimensional (or two-dimensional) measure zero lead to singularity of the boundary-value problem. Allowance for these conditions in traditional mathematical methods of elasticity theory is therefore a fairly complicated problem. The BEM implements these conditions by defining the zero values of the components of the displacement vector at the corresponding nodes of the discrete surface approximating the given surface. By circular symmetry, we triangulated one eighth of the spherical surface with an appropriate node indexing, which subsequently ensured fairly simple construction of the matrix of the DBE system with a minimum bandwidth. The isoparametric BE approximation for plane triangular BEs Δs_n is characterized by the fact [5, 8] that the interpolation functions ψ_k are identical to the local coordinates η_k, $k = 1, 2, 3$; thus, the approximations (2.1), (2.2) have the form

$$y_n^{(i)} = \sum_{k=1}^{3} y_{nk}^{(i)} \eta_k, \quad u_n^{(i)} = \sum_{k=1}^{3} U_{nk}^{(i)} \eta_k, \quad i = 1, 2, 3, \quad \forall n$$

where $y_{nk}^{(i)}$ are the Cartesian coordinates of the mesh nodes formed by the meridians and the parallels. The DBE are formed by a seven-point scheme (by a five-point scheme for bilinear BE approximation); each nodal value $U_{nk}^{(i)}$, $i = 1, 2, 3$ is multiplied by the sum of the contributions of six BEs; the resulting matrix is block-banded with bandwidth 7 and symmetric blocks. Given the symmetrically defined (on meridional lines) homogeneous boundary conditions for the displacement vector, the recommendations of [16] enable us to preserve the symmetry of the submatrices (blocks). The column vector on the right-hand side of the DBE system is constructed using the BE approximation of the $y^{(i)}$ axis components of the given normal load

$$p_i^{(\nu_n)} = l_n^{(i)} \sum_{k=1}^{3} p^{(\nu_n)} (y_{nk}^{(i)}) \eta_k, \quad i = 1, 2, 3, \quad \forall n$$

where $p^{(\nu_n)}(y_{nk}^{(i)})$ are the nodal values of the function $\mathbf{p}^{(\nu_n)}(y^{(i)})$.

To solve the DBE system, we followed the recommendations of [16]. In particular, the inversion procedure for symmetrical submatrices enabled us to implement the Gauss elimination method for reducing a square matrix consisting of submatrices to an upper triangular block matrix. A numerical experiment was carried out evaluating the sum (3.3): for $N = 15$, 30 and 60 we obtained, respectively, $\Sigma_N = 0.835$; 0.627; 0.353.

We considered the prototype problem of a normal tearing crack in an elastic unbounded plate loaded at infinity by a uniformly distributed stretching load p. The analytical solution of this problem is known (see, e.g. [17]) in the form of the displacement field $(u^{(1)}, u^{(2)})$ and the stress field $(\sigma^{(1)}, \sigma^{(2)}, \sigma^{(12)})$, whose components are defined as functions of polar coordinates (r, θ) (with the origin at the tip of the crack); the solution is exact (for small r) in a region ahead of the tip of the crack, which is much smaller than the crack width $2l$; the stress intensity factor [17] is $K_I = p\sqrt{\pi l}\sin^2\beta$, where β is the angle that the plane of the crack makes with the load axis.

For the variational problem (1.1) (the applicability of this problem was suggested in [18], see R. V. Gol'dshtein's supplement), we represented the plane of the crack in the Cartesian coordinate system $(y^{(1)}, y^{(2)})$, $y^{(1)}$ is the axis in the plane of the crack. We assumed that the crack contour $S = S_+ \cup S_-$ was traced by circular arcs of small curvatures and was the interior boundary of the region of the plate with a singular point at the tip of the crack. We considered a linear BE approximation of the form (2.1), (2.2); to approximate the given function $g(\nu)$, the nodal values $g_{nk}^{(i)}$, $i = 1, 2$ [see (2.5)] were defined in terms of the values of the components $\sigma^{(1)}$, $\sigma^{(2)}$, $\sigma^{(12)}$ at these nodes; the increase in the stresses in the neighbourhood of the singular points was allowed for by condensation of nodes, and the DBE were written for the nodes $k \in S_{+\Delta}$ and $k' \in S_{-\Delta}$ sufficiently close to the singular point. We compared the nodal values of the displacement components $U_{nk}^{(i)}$, $i = 1$, 2 obtained from the DBE system with the nodal values of the analytical solution $u^{(i)}$; with a quarter of S partitioned into $N = 6$, 12 and 24 boundary elements, the error averaged over the nodal points was $\varepsilon \cong 19$, 16 and 11%, respectively.

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THE PERTURBATION METHOD IN PROBLEMS OF THE DYNAMICS OF INHOMOGENEOUS ELASTIC RODS†

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The regular perturbation method (the small-parameter method) is developed in order to investigate the dynamics of weakly inhomogeneous rods with arbitrary distributed loads and boundary conditions of various types leading to self-conjugate boundary-value problems. The approach rests on the introduction of a perturbed argument, namely, the Euler variable, and a suitable representation of the eigenfunctions. It enables one to carry out uniform constructions of the basis and the eigenvalues, as well as the frequencies with any required accuracy in terms of the small parameter using quadratures of known functions. To illustrate the effectiveness, an example involving inhomogeneous rods with hinged left-hand ends and free right-hand ends and with box-shaped and circular cross-sections whose dimensions depend linearly on the coordinate are investigated and computed.

1. FORMULATION OF THE PROBLEM

CONTROLLED planar motions of an elastic rod undergoing transverse bending deformations are considered. Longitudinal extension will be neglected. It is assumed that the neutral line of the unstrained rod is straight and the elastic strains are small, i.e. the motion of the rod can be described

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